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COMPUTATIONAL DIVERSIONS

# Computational Diversions: Turtle *Really and Truly* Escapes the Plane

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> "But how do I get there?" asked Boots. "That's easy," the old man replied. "Just put one foot before the other and follow your nose."
> – Fairy tale [Quoted in (Abelson and diSessa 1980), p. 201].

Just about anyone, when asked, can respond to the question "What is your all-time favorite book?" I'm no exception–in fact, for me, there's no contest. The book is *Turtle Geometry*, by Hal Abelson and Andy diSessa (Abelson and diSessa 1980). Actually, *Turtle Geometry* is more than just a "favorite" book: it actually changed my life, a story I've never written about. In 1980, while working as a fledgling computer programmer at the Rockefeller University (my machine there was a DEC PDP-8), I happened into the University Library and plucked the book from the "new arrivals" shelf, intrigued by the mysterious title. A month later, I was certifiably insane, telling everyone who would listen that there was this *really, really important book* that *you just had to read*, that introduced mathematical ideas in a *completely wonderful, experimental way*, that made *everything clear*… and so forth. I had no idea who the authors were, but the book jacket said that they worked at MIT, so I applied to graduate school to study with them. My application form must have appeared sufficiently obsessive, and to my shock I was accepted. It sounds arrogant to say that "the rest is history", but in any event, the rest is *my* history.

I still love *Turtle Geometry*, and have probably re-read the book in its entirety a half dozen times in the three decades since it was published. Abelson and diSessa use a Logo-like language to present a remarkably accessible, procedural approach to geometry. In the course of about 500 pages, they introduce ideas of recursion, artificial life, Euclidean geometry, vector algebra, topology, and general relativity, to name a few. The book has lost absolutely none of its joy and freshness for me; by the way, it's still in print, and if the interested reader wishes to consider this a plug, he or she can be my guest.

The purpose of this column is to quench an ambition that has festered within me these past 30 years, since first picking up *Turtle Geometry* from the new-arrivals shelf. In the

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fifth chapter of the book, titled "Turtle Escapes the Plane", Abelson and diSessa describe the idea of a Logo turtle moving about, not on a planar surface (as in most Logo programming), but on the surface of a sphere. They begin the discussion with a striking example of a triangle containing three right angles–a manifest impossibility in the plane: (Fig. 1)

> Imagine that a turtle is crawling on a sphere–the earth's surface, for example… Starting at the equator and facing north, the turtle goes straight north until it reaches the north pole. There it turns 90° and goes straight south until it gets to the equator. Again it turns 90° and runs along the equator to get back to its initial position, where a final 90° turn restores its initial heading. (Abelson and diSessa 1980, p. 202)

The three-right-angle triangle demonstrates that spherical geometry doesn't work according to the same rules as Euclidean geometry. While there isn't sufficient space here for a full presentation (the reader is referred to *Turtle Geometry*, Chap. 5 for that), we can outline the major themes of spherical turtle geometry. First, we note that on the sphere, "straight lines" for the Logo turtle are the arcs of great circles. (Thus, for example, in the triangle walk just described, the turtle crawled along three great circles–namely, two longitude lines and the equator. As an aside, we note that with the exception of the equator, latitude lines are *not* "straight turtle lines", since they are not great circles.) Moreover, the interior angles of any spherical triangle depend on the enclosed area of the triangle. Consider, for example, our sample triangle–which we note is an equilateral triangle, with three equal (one-quarter circumference) sides. The interior angles of this triangle total 270°, as we have just seen; but by contrast, a very tiny equilateral triangle will look almost like our familiar planar variety and will have interior angles totaling just a bit more than 180°. In fact, as Abelson and diSessa explain, the excess interior angle total of a spherical triangle–that is, the amount by which the interior angle total exceeds the "planar standard" of 180°–is directly proportional to the enclosed area. The formula expressing the "excess angle" is:

$$E = A \times 720^{\circ}$$

where A is the area of the spherical triangle measured on a sphere whose total surface area is equal to 1. Thus, our first sample triangle, which covers one-eighth of the sphere's surface, has an extra

$$720/8 = 90$$

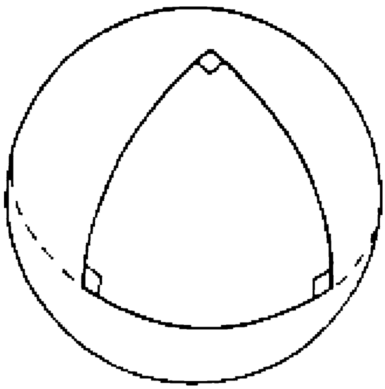

**Fig. 1** A triangle with three right angles, made by the spherical turtle walk described in the text (Picture originally in (Abelson and diSessa 1980), p. 202)

degrees in the total of its interior angles. A smaller triangle–covering, say, one hundredth of the sphere's surface–would have an interior excess of 7.2°, and would thus have interior angles totaling 187.2°.

All of this is fun and interesting, but reading *Turtle Geometry* back in 1980 I found myself a bit frustrated–there was really no way to *see* the turtle actually moving on a sphere. Certainly, it would be possible (in principle, at least) to create a program that would display a flat-screen-based simulation of a turtle moving on a sphere–though admittedly, this sort of project was beyond my powers as a programmer back then. Years later, I did manage to write a package displaying (on a computer screen) a portrait of a sphere with a turtle moving about on its surface, and thereby was able to experiment directly with the examples from the book. Even in this case, however, the result was only a *picture* of a sphere, not the real thing.

Well, I am now delighted to say that I have seen, and played with, the real thing. Thanks to six students at the University of Colorado–graduate student Michael MacFerrin and undergraduates James Bailey, Brian Hallesy, Neal Robbins, Garett Shulman, and Brandon Shelton–the Logo turtle can really and truly escape the plane and journey on a sphere. The students have implemented a working prototype of a Logo turtle package that runs at our University's Fiske planetarium, on the giant "Science on a Sphere" display in the main lobby. The photograph in Fig. 2 shows an example of a turtle walk displayed on the sphere–in this case, three complete great-circle walks that intersect at right angles.

A few points are worth noting before going any further. First, a word or two about the spherical display itself. The "Science on a Sphere" display was developed at the National Oceanic and Atmospheric Administration (NOAA) by Dr. Alexander MacDonald as an educational device for viewing computer-generated graphics on a sphere. The NOAA website (NOAA website for "Science on a Sphere": http://sos.noaa.gov/index.html) has many more details on the device, but for brevity I'll quote that site for a summary description:

> Science On a Sphere® is a large visualization system that uses computers and video projectors to display animated data onto the outside of a sphere. Said another way, SOS is an animated globe that can show dynamic, animated images of the atmosphere, oceans, and land of a planet. NOAA primarily uses SOS as an education and outreach tool to describe the environmental processes of Earth. (NOAA website for "Science on a Sphere": http://sos.noaa.gov/index.html).

There are, by now, numerous sphere displays installed at museums around the globe. The impressive size of the display can be inferred from the photograph in Fig. 2: the effect of seeing a high-resolution portrait of (say) the surface of the moon on the display is truly

**Fig. 2** Three intersecting great circles, drawn by a spherical Logo turtle on the Fiske Planetarium "Science on a Sphere" display in Boulder, Colorado

stunning. An image or video search on the Web for "Science on a Sphere" (e.g. at sites like YouTube, Flickr, and Vimeo) will turn up numerous compelling examples of its use; I should mention, though, that at least to my knowledge, the six young men at the University of Colorado are the first to develop an interactive turtle package for the display.

Now, a word about the particular example of Fig. 2. The reader will note that the "three-right-angle triangle" appears, front and center in the photograph, as the result of the intersection of these three great circles. This particular pattern was made by the following series of spherical turtle commands:

```
forward 360   ;this moves the turtle fully around the globe from its
              ;starting position. The circumference of the sphere
              ;is taken to be 360 units
right 90
forward 450   ;this makes a second complete great circle and moves
              ;the turtle fully around the globe one-and-a-quarter
              ;times
right 90
forward 360   ;this draws the third great circle
```

The reader should try to visualize the effects of this spherical program to see how it conforms to the pattern shown in the photograph of Fig. 2. The program will produce a similar pattern no matter where the turtle happens to be initially positioned on the sphere. The effect of a "right" turn is just as it is in standard Logo turtle graphics—i.e., the turtle turns 90° to the right in its present position. The only difference here is in the meaning of "forward", in which the turtle, obeying the fairy tale injunction quoted at the beginning of Chap. 5 in *Turtle Geometry* (and quoted here again at the outset of this column) simply takes equal-sized steps with its right and left legs and "follows its nose" along a great circle path.

A little reflection on the Fig. 2 pattern also reveals another interesting fact: the lines on the sphere are actually the projections onto the spherical surface of a regular octahedron inscribed within the display. To put it another way: imagine an octahedron (it would have to be a giant-sized octahedron) sitting within the sphere, with its six vertices just touching the surface. Then, if we were to project the edges of the octahedron outward onto the sphere's surface, we would see the pattern of lines visible in Fig. 2. An interesting aside here is to note that each of the crossing points in Fig. 2 corresponds to a vertex of the octahedron; consequently, on the sphere, it is possible that four *equilateral* triangles can surround a single point. (Just try doing that on a plane).

Emboldened by this example, we might now ask whether we could show, on the sphere, patterns generated by the edges of other solids. The octahedron, consisting of eight equilateral triangles, is one of five regular (or "Platonic") solids: the others are the cube (six squares), tetrahedron (four equilateral triangles), dodecahedron (twelve regular pentagons), and icosahedron (twenty equilateral triangles). Can we show the edge patterns of these other solids when inscribed in the sphere?

The affirmative answer is demonstrated in Figs. 3 and 4. Figure 3 shows sketches of the five Platonic solids; Fig. 4 shows photographs of their projections onto the surface of the sphere (with the exception of the octahedron, which we've already produced in Fig. 2).

The programs that generated these figures would be too long to reproduce here (and in any event, it must be confessed that they were written in an excited rush and weren't particularly elegant). Still, it's worth pointing out the necessity, in each case, of figuring out what side length will be needed to produce the appropriate line segments on the sphere.

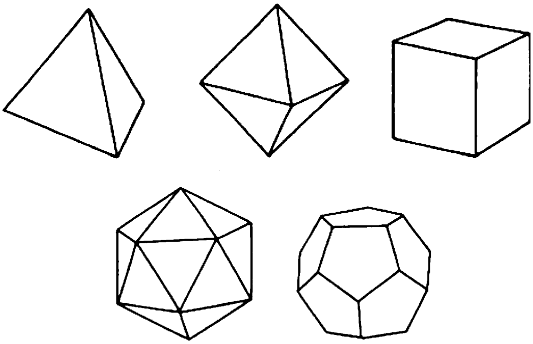

**Fig. 3** The five Platonic solids. *Top row* tetrahedron, octahedron, cube. *Bottom row* icosahedron, dodecahedron. These are the only five solids with the following properties: all faces are regular polygons, all faces have the same number of edges, all edges are the same length, and all vertices are surrounded by equal numbers of faces

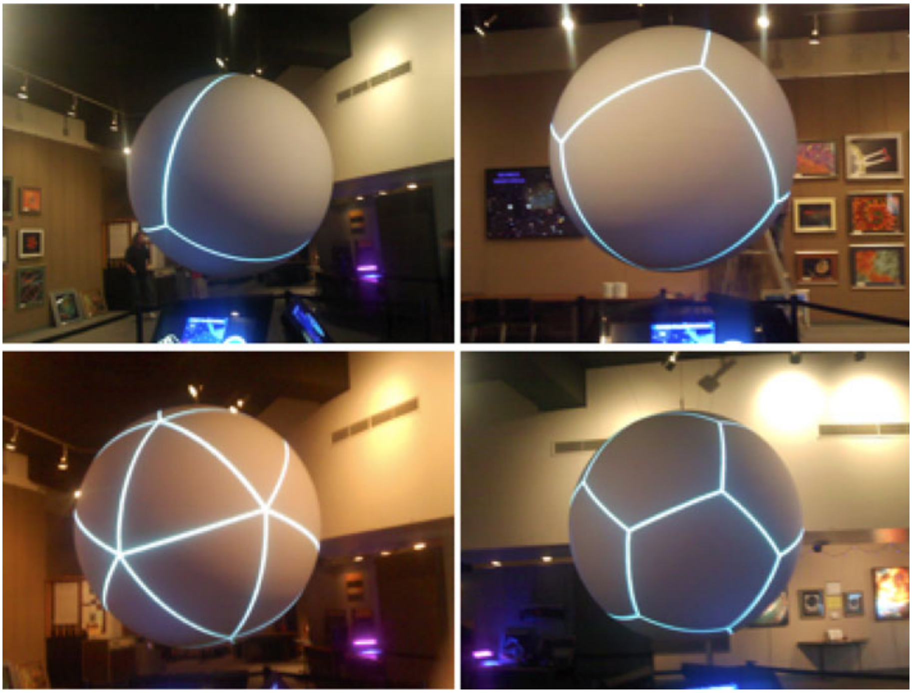

**Fig. 4** The inscribed tetrahedron, cube, icosahedron, and dodecahedron shown on the spherical display, as created by spherical Logo turtle walks

Consider, for example, the embedded cube shown at the upper right of Fig. 4. Each square is generated by a spherical Logo turtle walk of the form:

```
repeat 4
 forward distance
 right 60
```

Of course, to the practiced Logo-programmer eye, this pattern looks odd: shouldn't a square be generated by right 90 turns, not right 60 turns? And the answer again is that things work differently on the sphere. Note that at each vertex of the inscribed cube, three regular quadrilaterals (I suppose they could be called "spherical squares") meet. Thus, each interior angle at the vertex is 120°.

We're agreed, then, that the right turn in our program for generating the cube should be 60, and not 90. But what should the value of "distance" be? How far around a great circle should the turtle walk to make one edge of a square, each of whose interior angles is 120°? Looking at the cube photograph in Fig. 4, it appears that the distance walked along a single edge is not quite a quarter of the circumference of the sphere–i.e., the value of "distance" appears to be less than 90. But what, exactly, should it be?

To be perfectly honest, I found these numbers just by noodling directly with the turtle interface: that is, I tried drawing a bunch of squares on the sphere until I found the correct side-length for a square with a 120° interior angle. But it's certainly possible to find the value in a more dignified fashion. Imagine a cube of side-length 2, centered at the origin. Consider two of the adjacent vertices of this cube: say, (1, 1, 1) and (1, 1, −1), and imagine vectors drawn from the center of the cube to each of these two vertices. What we want to know is the angle between these two vectors–that is, the angle between the vectors starting at the origin and ending at (1, 1, 1) and (1, 1, −1), respectively. This angle will tell us the extent around the great circle that we need to traverse in order to make the cube pattern at the upper right of Fig. 4.

The angle between the two vectors can be found by taking the dot product of the two vectors; once this dot product is known, we then make use of the formula relating the dot product to angle:

$$\mathbf{V} \bullet \mathbf{V}' = (\text{length } \mathbf{V}) \times (\text{length } \mathbf{V}') \times \cos(\text{angle between } \mathbf{V} \text{ and } \mathbf{V}')$$

The dot product between our two vectors is just $(1 + 1 - 1) = 1$, and the length of the two vectors are each square root of 3. Thus, to find the angle $\Theta$ between the two vectors, we use the formula given above to find:

$$1 = 3 \cos \Theta$$

The value of $\Theta$ for which the cosine is 0.333 is about 70.5°; so this is the distance that the turtle needs to travel to produce a side of the inscribed cube. In short, then, we have found the recipe for an inscribed-cube square on the sphere (again using 360 as the distance around the circumference):

```
repeat 4
  forward 70.5
  right 60
```

Similar calculations can be done for each of the Platonic solids. One needs to find the coordinates of the vertices of the solids when centered at the origin; and then (using the dot product), find the angle between vectors starting at the origin and ending at adjacent vertices.

Clearly, there are innumerable projects still to explore with this wonderful display–maybe enough for an entire follow-up volume to Abelson and diSessa's original book. Moreover, the NOAA spherical display is itself just one of a host of display innovations–including three-dimensional display screens, volumetric displays, and flexible display screens–that could be profitably combined with Abelson and diSessa's ideas (and with creative mathematics education more generally). That is a subject for a much longer and more detailed discussion. Still, I cannot conclude this column without including one valedictory tribute to the wonderful influence of *Turtle Geometry*: a Logo flower drawn on the

**Fig. 5** A happy-thirtieth-anniversary spherical flower for *Turtle Geometry*

sphere. This can be considered as a “bouquet”, three decades in the making, and happily dedicated to the two authors. Readers are encouraged to send in their own spherical turtle projects, or suggestions for spherical projects, as well, using the email address for this column (Fig. 5).

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## References

Abelson, H., & diSessa, A. (1980). *Turtle Geometry*. Cambridge, MA: MIT Press.